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Möbius–deBruijn: The product of Möbius cube and deBruijn digraph ☆

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ABSTRACT

Möbius cube and deBruijn digraph have been proved to be two of the most popular interconnection architectures, due to their desirable properties. Some of the attractive properties of one, however, are not found in the other. The Möbius–deBruijn architecture, proposed in this paper, is the product of Möbius cube and deBruijn digraph, which is a combination of the two architectures. It employs the Möbius cube as a unit cluster and connects many such clusters by means of given number of parallel deBruijn digraphs. Consequently, the Möbius–deBruijn provides some of the desirable properties of both the architectures, such as the flexibility in terms of embedding of parallel algorithms, the high level of fault-tolerant, and the efficient inter-cluster communication. The proposed architecture also possesses the logarithmic diameter, the optimal connectivity, and the simple routing mechanism amenable to network faults. The methodology to construct the Möbius–deBruijn can apply to the product of deBruijn digraph and other hypercube-like networks, and also applies to the product of Kautz digraph and hypercube-like networks.

1. Introduction

Hypercube has been proved to be one of the most popular architectures, due to its attractive properties such as strong connectivity, regularity, topological symmetry, and recursive constructions. Many efforts have been done to give better performance by adding extra links and thus reducing the network diameter of hypercube. On the contrary, many variants of hypercube have been proposed to reduce the network diameter with the same number of links and nodes after simply rearranging all links. The Möbius cube is one of such hypercube variants and its diameter is the minimum one that can be obtained with these resources [1]. As shown in Fig. 1, the 3-dimensional Möbius cube is of diameter 2. The 3-dimensional hypercube, however, requires being added additional 4-links so as to be of diameter 2.

The node degree of hypercube and its variants grows logarithmically with respect to the order of the network. Although this property provides the high level of fault tolerance, the complexity of the node structure becomes

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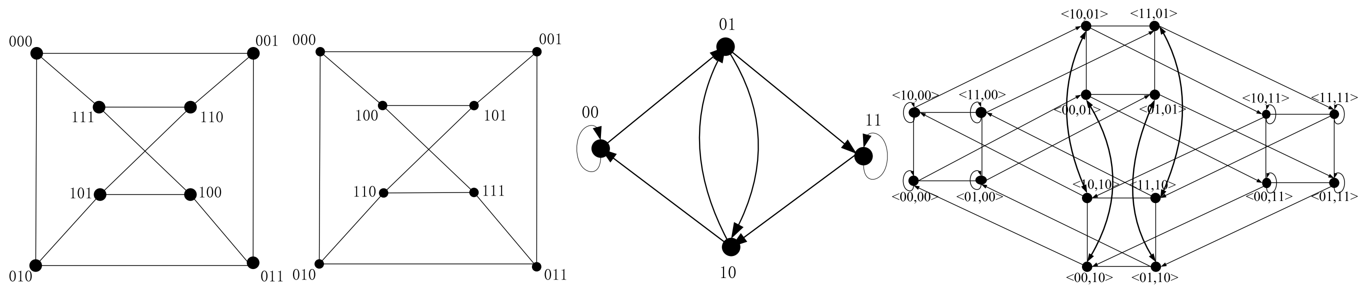

(a) 1-Möbius cube $M(3)$. (b) 0-Möbius cube $M(3)$. (c) deBruijn digraph $D(2,2)$. (d) The Möbius-deBruijn $MD(2,2,2)$.

Fig. 1. Illustrative examples of $M(3)$, $D(2,2)$, and $MD(2,2,2)$.

prohibitive for large networks. On the other hand, the node degree remains fixed, regardless of the network size, for constant degree networks, such as cube-connected-cycle (CCC) [2], butterfly [3], deBruijn [4], and Kautz digraph [5,6]. One shortcoming of such networks is that the level of fault tolerance does not grow with the size of network, though this requirement is great important in practice. Therefore, it is desirable to design new architectures, which combine the low port requirement of constant degree networks with the desired level of fault tolerance of hypercube-like networks.

Inspired by this motivation, this paper proposes a new class of architecture called Möbius–deBruijn, which is the product of Möbius cube and deBruijn digraph. It employs the Möbius cube as a unit cluster and connects many such clusters by means of given number of parallel deBruijn digraphs. Consequently, the Möbius–deBruijn incarnates the desirable properties of both the Möbius cube and the deBruijn digraph, such as the flexibility in terms of embedding of parallel algorithms, the high level of fault-tolerant, and efficient inter-cluster communication. The proposed architecture also possesses the logarithmic diameter, the optimal connectivity, and the simple routing mechanism amenable to network faults.

Among the state-of-the-art schemes, the hyper–deBruijn is closest to our work in this paper. Hyper–deBruijn is the product of undirected deBruijn graph and hypercube. The construction and routing methodologies of the hyper–deBruijn, however, cannot apply to the Möbius–deBruijn. To the best of our knowledge, this is the first work dealing with the product of deBruijn digraph and hypercube-like networks. Möbius–deBruijn provides a general construction methodology due to the following reasons. Generally, the Möbius–deBruijn is just the product of deBruijn digraph and hypercube if the dimension of the Möbius cube does not exceed two. Although this paper focuses the product of deBruijn digraph and Möbius cube, the basic ideas, however, apply to the product of deBruijn digraph and other hypercube-like networks with the similar diameter as the Möbius cube, such as twisted cube [7], flip MCube [8], and fastcube [9]. Similarly, the basic ideas can also be used to support the product of Kautz digraph and any of those hypercube-like networks. Moreover, compared to the product of deBruijn digraph and hypercube, the Möbius–deBruijn generates a network with smaller diameter under the same node degree and network size. In addition, compared to hyper–deBruijn, the same result holds if we replace the deBruijn digraph with an undirected one for the Möbius–deBruijn architecture.

2. Möbius–deBruijn networks

Let the interconnection network be modeled by a graph/digraph $G(V,E)$, where the set of vertices V, represents the processors in the network and the set of edges E, represents the communication links in the network. In the rest of this paper, we use the terms network and graph/digraph, node and vertex, link and edge, interchangeably.

2.1. Notation and definitions

The m-dimensional *hypercube* is denoted by $H(m)$, where $m \geqslant 1$. The vertex set of $H(m)$ is $\{x_m \dots x_i \dots x_1\}$, where $x_m \dots x_i \dots x_1$ denotes a sequence and $x_i = 0$ or 1 for any $1 \leqslant i \leqslant m$. There is an edge between any two vertices if and only if their labels differ by exactly one bit. There are 2^m vertices in $H(m)$. The node degree and network diameter are m.

The m-dimensional Möbius cube [10] is denoted by $M(m)$. The vertex set of $M(m)$ is the same as that of $H(m)$. For any vertex $X = x_m \dots x_2 x_1$, it connects to m other vertices Y_j $(1 \leqslant j \leqslant m)$, where Y_j satisfies one of the following equations:

$$Y_j = \begin{cases} x_m \dots x_{j+1}\overline{x_j}x_{j-1} \dots x_1, & \text{if } x_{j+1} = 0 \\ x_m \dots x_{j+1}\overline{x_j x_{j-1} \dots x_1}, & \text{if } x_{j+1} = 1 \end{cases} \tag{1}$$

The connection between vertices X and Y_m has x_{m+1} undefined. x_{m+1} is neither equal to 1 or 0, which results in slightly different network topologies. If $x_{m+1} = 1$, the resulting network is called a 1-Möbius cube, as shown in Fig. 1(a). Otherwise, it is a 0-Möbius cube, as shown in Fig. 1(b). The node degree of the resulting network is m, irrespective the value of x_{m+1}. The network diameters of the 1-Möbius cube and 0-Möbius cube are $\lceil (m+1)/2 \rceil$ and $\lceil (m+2)/2 \rceil$, respectively. It is worthy noticing that the m-dimensional Möbius cube is just the hypercube, when $m \leqslant 2$.

The d-ary k-dimensional deBruijn digraph [4,11], $D(d,k)$, has node out-degree of d and network diameter of k, where $d \geqslant 1$ and $k \geqslant 1$. The vertex set is $\{x_k \dots x_i \dots x_1 \mid x_i \in \{0,1,\dots,d-1\}$ for all $1 \leqslant i \leqslant k\}$. There is an arc from a vertex $x_k x_{k-1} \dots x_1$ to a vertex $x_{k-1} \dots x_1 \alpha$ for each $\alpha \in \{0,1,\dots,d-1\}$. The arc is said to be incident from the

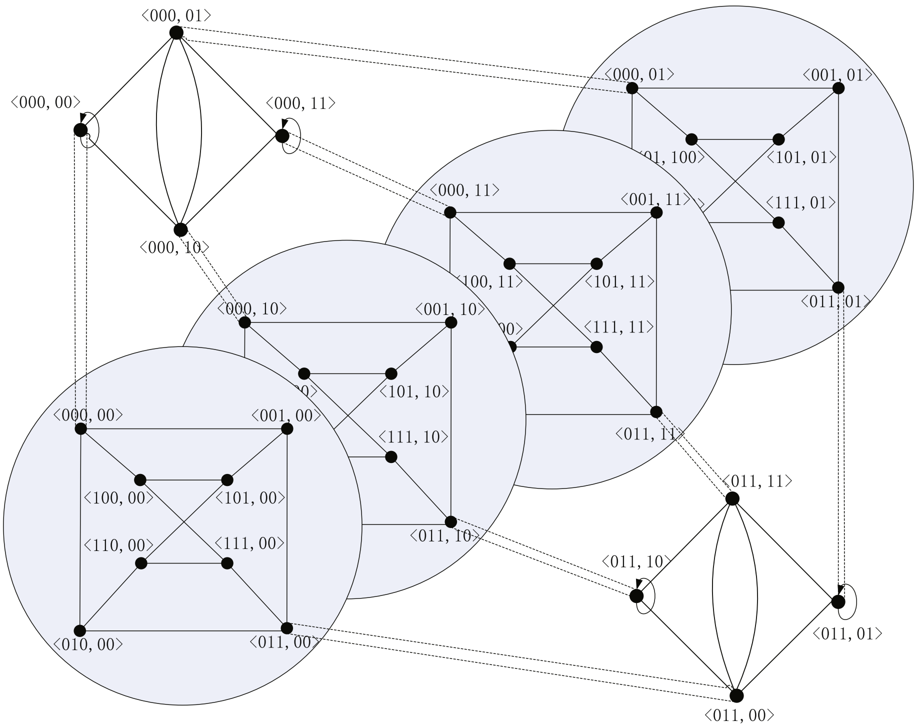

Fig. 2. The Möbius–deBruijn $(3, 2, 2)$.

vertex $x_k x_{k-1} \dots x_1$ and incident on the vertex $x_{k-1} \dots x_1 \alpha$. The set of vertices incident from the vertex $x_k x_{k-1} \dots x_1$ is called its out-neighbors, while the set of vertices incident on the vertex $x_k x_{k-1} \dots x_1$ is called its in-neighbors. There are d^k vertices in $D(d, k)$. The in-degree and out-degree of any node are the same d. Fig. 1(c) illustrates an example of the 2-ary 2-dimensional deBruijn digraph $D(2, 2)$.

2.2. Construction methodology of Möbius–deBruijn

Given two graphs G_2 and G_1, the product $G = G_2 \times G_1$ is defined as follows. The node set V of G, is the Cartesian product $V_2 \times V_1 = \{\langle v_2, v_1\rangle \mid v_2 \in V_2 \text{ and } v_1 \in V_1\}$, where V_2 and V_1 denote the node set of G_2 and G_1, respectively. The label of any node in G is a concatenation of two address fields with one each from the two basic graphs. Two nodes (u, x) and (v, y) of G are adjacent if and only if either (u, v) is an edge of G_2 and $x = y$ or (x, y) is an edge of G_1 and $u = v$. It is easy to see that $G_2 \times G_1$ is isomorphic to $G_1 \times G_2$ since there is a one-to-one correspondence between their sets of nodes which preserves the adjacency.

Definition 1. The Möbius–deBruijn $MD(m, d, k)$ is defined as the product graph of the deBruijn digraph $G_2 = D(d, k)$ and the Möbius cube $G_1 = M(m)$.

The Möbius–deBruijn can be constructed through one of the following two ways. In the first way, it employs the Möbius cube as a unit cluster and connects $|V_2| = d^k$ such clusters by means of $|V_1| = 2^m$ deBruijn digraphs. In the resultant graph, there is $2 \times d$ additional remote links associated with each node in the Möbius cube. In the second way, it utilizes the deBruijn digraph as a unit cluster and connects $|V_1| = 2^m$ such clusters by means of $|V_2| = d^k$ Möbius cubes. In the resultant graph, there is m additional remote links associated with each node in the deBruijn digraph.

We further present the formal definition of the Möbius–deBruijn $MD(m, d, k)$ as follows.

Definition 2. For any node $\langle x_m \dots x_2 x_1, y_k \dots y_2 y_1\rangle$ in $MD(m, d, k)$, it has an out-arc to a node $\langle x_m \dots x_2 x_1, y_{k-1} \dots y_1 \alpha\rangle$ and an in-arc from a node $\langle x_m \dots x_2 x_1, \alpha y_k \dots y_2\rangle$, for each $\alpha \in \{0, 1, \dots, d-1\}$. These nodes are referred to as the deBruijn-part-neighbors of the node $\langle x_m \dots x_2 x_1, y_k \dots y_2 y_1\rangle$, which also has an edge to each of the following nodes referred to as the Möbius-part-neighbors

$$\begin{aligned} &\langle x_m \dots x_{j+1} \overline{x_j} x_{j-1} \dots x_1, y_k \dots y_2 y_1\rangle, \quad \text{if } x_{j+1} = 0 \\ &\langle x_m \dots x_{j+1} \overline{x_j x_{j-1} \dots x_1}, y_k \dots y_2 y_1\rangle, \quad \text{if } x_{j+1} = 1 \qquad (2) \end{aligned}$$

for all j, $1 \leqslant j \leqslant m$.

Given a node $\langle x, y\rangle$, we will call x as its Möbius-part-label and y as its deBruijn-part-label. All nodes with the same deBruijn-part-label form an m-dimensional Möbius cube, while nodes with the same Möbius-part-label form a d-ary k-dimensional deBruijn graph. Actually, there are d^k Möbius cube subgraphs $M(m)$ in $MD(m, d, k)$, where nodes in each $M(m)$ have the same deBruijn-part-label. We will represent these subgraphs by their corresponding deBruijn-part-labels. Similarly, $MD(m, d, k)$ can be visualized as having 2^m deBruijn subgraphs $D(d, k)$ where nodes in each $D(d, k)$ have the same Möbius-part-label. We will represent these subgraphs by their corresponding Möbius-part-labels.

Fig. 2 shows a representation of Möbius–deBruijn. We can see that there are $d^k = 2^2$ Möbius cube subgraphs $M(3)$ and $2^m = 2^3$ deBruijn subgraphs $D(2, 2)$ in

Algorithm 1 Routing algorithm in $MD(m,d,k)$

Require: Let $\langle x,y\rangle$ and $\langle u,v\rangle$ be the source node and destination node, respectively.

1: Let *leftpath* denote the routing path from the node $\langle x,y\rangle$ to the node $\langle x,v\rangle$. It is easy to see that the two nodes locate in the same deBruijn subgraph since they hold the same Möbius-part-label. For each intermediate node in *leftpath*, its Möbius-part-label is x, while its deBruijn-part-label can be obtained by the routing algorithm of the deBruijn digraph [4].

2: Let *rightpath* denote the routing path from the node $\langle x,v\rangle$ to the node $\langle u,v\rangle$. It is easy to see that the two nodes locate in the same Möbius cube since they hold the same deBruijn-part-label. For each intermediate node in *rightpath*, its deBruijn-part-label is v, while its Möbius-part-label can be achieved using the routing algorithm of the Möbius cube [10].

3: The routing path from the node $\langle x,y\rangle$ to the node $\langle u,v\rangle$ is a concatenation of *leftpath* and *rightpath*.

$MD(3,2,2)$. Note that only two deBruijn subgraphs labeled 000 and 011 are plotted, while other six deBruijn subgraphs labeled 001, 010, 100, 101, 110, and 111 are not shown due to page limitations. For ease of presentation, we illustrate an example of $MD(2,2,2)$ in Fig. 1(d). Note that $MD(2,2,2)$ is the product of 2-dimensional hypercube and 2-ary 2-dimensional deBruijn digraph.

2.3. Routing in Möbius–deBruijn networks

For Möbius–deBruijn networks, routing should be simple and fast so as to ensure that any message can be forwarded from a source node $\langle x,y\rangle$ to a destination node $\langle u,v\rangle$ efficiently. For a fault-free Möbius–deBruijn network, routing can be implemented by Algorithm 1, which systematically integrates the existing routing algorithms for Möbius cube and deBruijn digraph.

For example, a message from a node $\langle 10,00\rangle$ to a node $\langle 01,11\rangle$ is first routed to a node $\langle 10,11\rangle$ along *leftpath*, resulting from the first step of Algorithm 1, as shown in Fig. 1(d). Here, *leftpath* is denoted as $\langle 10,00\rangle \to \langle 10,01\rangle \to \langle 10,11\rangle$. That message is then forwarded to node $\langle 01,11\rangle$ along *rightpath*, resulting from the second step of Algorithm 1. Here, *rightpath* is one of the two possible paths $\langle 10,11\rangle \to \langle 11,11\rangle \to \langle 01,11\rangle$ or $\langle 10,11\rangle \to \langle 00,11\rangle \to \langle 01,11\rangle$.

It is worthy noticing that the aforementioned routing for fault-free Möbius–deBruijn network can be implemented in another way. Given a source node $\langle x,y\rangle$ and a destination node $\langle u,v\rangle$, let *leftpath* denote the routing path from node $\langle x,y\rangle$ to node $\langle u,y\rangle$, coming from the routing algorithm of the Möbius cube, while *rightpath* denote that from node $\langle u,y\rangle$ to node $\langle u,v\rangle$, resulting from the routing algorithm of the deBruijn digraph. The two routing approaches can be used to identify node-disjoint paths for any pair of nodes, as discussed later.

2.4. Topology properties

Table 1 summarizes parameters of the Möbius cube, deBruijn digraph, and Möbius–deBruijn.

Theorem 1. *In a $MD(m,d,k)$, there are $d^k \times 2^m$ vertices. The degree of each node is $m+2d$.*

Proof. For any given node $\langle x,y\rangle$, according to Definition 2, it follows that the number of nodes in $MD(m,d,k)$ is $2^m \times d^k$ since the number of its possible deBruijn-part-labels is d^k while that of the possible Möbius-part-labels is 2^m. From Definition 2, it follows that there is an arc from the node $\langle x,y\rangle$ to each of d deBruijn-part-neighbors and an arc from each of other d deBruijn-part-neighbors to node $\langle x,y\rangle$. Thus, the in-degree and out-degree of the node $\langle x,y\rangle$ become d, respectively. Recall that the node $\langle x,y\rangle$ also connect with m Möbius-part-neighbors using undirected links. Thus, the degree of any node in $MD(m,d,k)$ is $m+2d$. Thus, Theorem 1 holds. □

Theorem 2. *If a $M(m)$ is a 1-Möbius cube, the diameter of a $MD(m,d,k)$ is $\lceil \frac{m+1}{2}\rceil + k$. If a $M(m)$ is a 0-Möbius cube, the diameter of $MD(m,d,k)$ is $\lceil \frac{m+2}{2}\rceil + k$.*

Proof. Consider two arbitrary nodes $\langle x,y\rangle$ and $\langle u,v\rangle$ in $MD(m,d,k)$. Starting from the node $\langle x,y\rangle$, we can reach the node $\langle u,y\rangle$ by traversing a Möbius cube $M(m)$, which is formed by all the nodes with the same deBruijn-part-label y. It is easy to see that the shortest path from the node $\langle x,y\rangle$ to the node $\langle u,y\rangle$ is at most $\lceil (m+1)/2\rceil$ and $\lceil (m+2)/2\rceil$ hops for the cases of 1-Möbius and 0-Möbius, respectively. In addition, proceeding from the node $\langle u,y\rangle$, we can reach the node $\langle u,v\rangle$ by traversing a deBruijn graph $D(d,k)$, which is formed by all the nodes with the same Möbius-part-label u. It is clear that the shortest path from the node $\langle u,y\rangle$ to the node $\langle u,v\rangle$ is at most k hops. Therefore, we can reach the node $\langle u,v\rangle$ from the node $\langle x,y\rangle$ in at most $\lceil (m+1)/2\rceil + k$ and $\lceil (m+2)/2\rceil + k$ hops for the cases of 1-Möbius and 0-Möbius, respectively. Hence Theorem 2 holds. □

A set of paths is said to be node-disjoint if no intermediate node appears in more than one path. Given any pair of nodes, the number of node-disjoint paths between them is an important property of any interconnection network. If such paths exist, they can be fully utilized to enhance the transmission reliability by adaptively choosing alternative paths when a given path fails, or to accelerate the transmission of data between any two nodes. Thus, the number of node-disjoint paths is a measurement of the level of fault tolerance for any interconnection network.

Table 1
Parameters of the networks of interest.

Graph	Network size	Number of edges	Degree	Diameter	Fault tolerance
1-Möbius $M(m)$	$N_1 = 2^m$	$L_1 = m2^{m-1}$	m	$d_1 = \lceil (m+1)/2\rceil$	$m-1$
0-Möbius $M(m)$	$N_1 = 2^m$	$L_1 = m2^{m-1}$	m	$d_1 = \lceil (m+2)/2\rceil$	$m-1$
$D(d,k)$	$N_2 = d^k$	$L_2 = d^{k+1}$	$2d$	$d_2 = k$	$d-2$
$MD(m,d,k)$	$N_1 \times N_2 = 2^m \times d^k$	$N_1L_2 + N_2L_1 = 2^m d^{k+1} + md^k 2^{m-1}$	$2d+m$	$d_1 + d_2$	$m+d-2$

Lemma 1. *(See [10].) In the m-dimensional Möbius cube $M(m)$, there are m node-disjoint paths between any pair of nodes if there are no faults. Therefore, $M(m)$ can tolerate up to $m-1$ node failures without disruption in communication between any pair of nodes.*

Lemma 2. *(See [4,12].) In the d-ary k-dimensional deBruijn digraph $D(d,k)$, there exist at least $d-1$ node-disjoint paths between any pair of nodes if there are no faults. Therefore, $D(d,k)$ can tolerate up to $d-2$ node failures without disruption in communication between any pair of nodes.*

Lemma 3. *For the second node $z = z_k \ldots z_2 z_1$ in any one among the $d-1$ node-disjoint paths from a node $x = x_k \ldots x_2 x_1$ to a node $y = y_k \ldots y_2 y_1$ in the $D(d,k)$, there exist cycles each of which has an inner vertex z and is of length at most $k+1$. The resultant cycles do not contain an inner vertex of those $d-1$ node-disjoint paths except the vertex z.*

Proof. Let $z_{k-1} \ldots z_1 z_u$ be the next hop of the node z along one of those $d-1$ independent pathes. The subpath from the node $z_k \ldots z_2 z_1$ to the node y is denoted as: $z_k \ldots z_2 z_1 \rightarrow z_{k-1} \ldots z_1 z_u \rightarrow z_{k-2} \ldots z_u y_i \rightarrow z_{k-3} \ldots z_u y_i y_{i-1} \rightarrow \cdots \rightarrow y_k \ldots y_2 y_1$. Note that i is determined as follows: (1) $i = k-1$ if $z_1 \neq y_k$ but $z_u = y_k$, for example $z = z_k \ldots z_2 z_1 = 002$ in the second path of Fig. 2; (2) $i = k-2$ if $z_1 z_u = y_k y_{k-1}$, for example $z = z_k \ldots z_2 z_1 = 001$ in the first path of Fig. 2.

Consider another out-neighbor $z_{k-1} \ldots z_1 z_v$ of the node $z_k \ldots z_2 z_1$, where $z_u \neq z_v$. A cycle can be constructed as follows: $z_k \ldots z_2 z_1 \rightarrow z_{k-1} \ldots z_1 z_v \rightarrow z_{k-2} \ldots z_v z_i \rightarrow z_{k-3} \ldots z_v z_i z_{i-1} \rightarrow \cdots \rightarrow z_k \ldots z_2 z_1$, where $i = k-1$ if $z_v = z_k$ or $i = k$ if $z_v \neq z_k$. Such cycle is of at most $k+1$ and of at least k hops. For the node 002 in the second node-disjoint path from the node 000 to the node 111, two cycles containing the node 002 are shown in Fig. 3.

The subpath proceeding from the node $z_k \ldots z_2 z_1$ to the node $y_k \ldots y_2 y_1$ does not contain any inner vertex of the resutling cycle except the node $z_k \ldots z_2 z_1$, due to the introduction of $z_u \neq z_v$. Moreover, this cycle does not traverse the source node $x_k \ldots x_2 x_1$, whose label is different from that of the previous node of the node $z_k \ldots z_2 z_1$ in that cycle at the leftmost bit. Thus, we can derive that this cycle does not intersect with the selected one path among those $d-1$ independent paths except the node $z_k \ldots z_2 z_1$.

Additionally, the node $z_k \ldots z_2 z_1$ does not present at other $d-2$ node-disjoint paths from the node $x = x_k \ldots x_2 x_1$ to the node $y = y_k \ldots y_2 y_1$. Consequently, the formed cycle does not intersect with other $d-2$ node-disjoint paths. Thus, Lemma 3 holds. □

Theorem 3. *In the Möbius–deBruijn $MD(m,d,k)$, there exist at least $m+d-1$ node-disjoint paths from a node $\langle x, y\rangle$ to a node $\langle u, v\rangle$ if there are no faults. Those node-disjoint paths are of length at most* $\max\{d_1 + 2k + 1, m\}$. *Therefore, a $MD(m,d,k)$ can tolerate up to $m+d-2$ node failures without disruption in communication between any pair of nodes.*

Proof. Since there are two classes of the Möbius cube, which differ in their network diameters, let d_1 denote the network diameter of the Möbius cube. The value of d_1 has been summarized in Table 1. To prove this theorem, there are three cases to be considered:

Case 1: $x \neq u$ and $y = v$. Consider the nodes with the same deBruijn-part-label v; they form a m-dimensional Möbius cube. We can find m node-disjoint paths within that Möbius cube from the node $\langle x, v\rangle$ to the node $\langle u, v\rangle$. Clearly, all nodes in such m paths have v as their deBruijn-part-labels.

Let $\langle x, v^{(i)} = v_{k-1} \ldots v_1 i\rangle$ for any $i \in \{0, 1, \ldots, d-1\}$ denote the deBruijn-part-neighbors of the source node $\langle x, v = v_k \ldots v_2 v_1\rangle$. Since nodes with the same deBruijn-part-label $v^{(i)}$ form a Möbius cube, proceeding from $\langle x, v^{(i)}\rangle$ we can reach $\langle u, v^{(i)}\rangle$ within that Möbius cube. Thus, for any given $0 \leqslant i \leqslant d-1$ we can find a path from the source node $\langle x, v\rangle$ to the node $\langle u, v^{(i)}\rangle$; this path is of length at most $d_1 + 1$ and does not intersect any of the previously formed m paths. It is worth noticing that nodes $\langle u, v^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-1$ and node $\langle u, v\rangle$ are within the same deBruijn subgraph. We further prove that there are at least $d-1$ node-disjoint paths from nodes $\langle u, v^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-1$ to the destination node $\langle u, v\rangle$ within that deBruijn subgraph.

According to the routing algorithm of deBruijn digraph, we can reach the node $\langle u, v\rangle$ from any node $\langle u, v^{(i)}\rangle$ along a path $\langle u, v^{(i)} = v_{k-1} \ldots v_1 i\rangle \rightarrow \langle u, v_{k-2} \ldots v_1 i v_k\rangle \rightarrow \langle u, v_{k-3} \ldots i v_k v_{k-1}\rangle \rightarrow \ldots \langle u, i v_k \ldots v_2\rangle \rightarrow \langle u, v = v_k \ldots v_2 v_1\rangle$. This path is of length at most k and does not intersect any of the previously formed m paths from the node $\langle x, y\rangle$ to the node $\langle u, v\rangle$ as well as those d paths from node $\langle x, y\rangle$ to nodes $\langle u, v^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-1$. For any two nodes $\langle u, v^{(i)}\rangle$ and $\langle u, v^{(j)}\rangle$ where $0 \leqslant i, j \leqslant d-1$ and $i \neq j$, the paths from that two nodes to the same destination node $\langle u, v\rangle$ are node-disjoint. Thus, the above construction yields other d node-disjoint paths from the source node $\langle x, y\rangle$ to the destination node $\langle u, v\rangle$, which are of length at most $d_1 + k + 1$. Consider the fact that one of the d out-neighbors is itself for some nodes $x_n \ldots x_2 x_1$ in a deBruijn digraph if $x_n = x_{n-1} = \cdots = x_2 = x_1$ for any $0 \leqslant x_1 \leqslant d-1$. Consequently, there are at most $m+d-1$ node-disjoint paths between any pair of nodes in a $MD(m,d,k)$. Since the previously formed m paths are of length at most m, those $m+d-1$ node-disjoint paths are of length at most $\max\{d_1 + k + 1, m\}$.

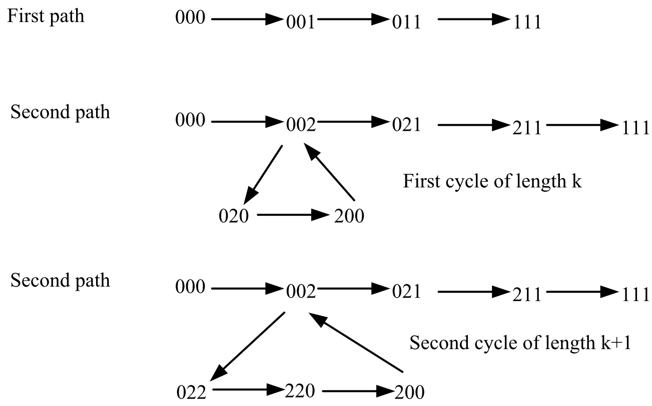

Fig. 3. An illustration example of node-disjoint paths between any pair of nodes in the deBruijn graph $D(3,3)$ and cycles defined in Lemma 3.

Case 2: $x = u$ and $y \neq v$. Consider the nodes with the same Möbius-part-label u; they form a deBruijn digraph $D(d,k)$. There exist $d-1$ node-disjoint paths from $\langle x, y\rangle$ to $\langle u, v\rangle$ within that deBruijn digraph since one deBruijn-neighbor of $\langle x, y\rangle$ might be itself. It is clear that those paths are of length at most k and all nodes in those paths have u as their Möbius-part-labels.

Assume that the Möbius-part-neighbors of the source node $\langle x = u, y\rangle$ are denoted as $\langle u^{(i)}, y\rangle$ for each $i \in \{1,\ldots,m\}$. Since the nodes with Möbius-part-label $u^{(i)}$ for any $1 \leqslant i \leqslant m$ form a deBruijn digraph, proceeding from $\langle u^{(i)}, y\rangle$ we can reach $\langle u^{(i)}, v\rangle$ through a path within this deBruijn digraph. Thus, for any given $1 \leqslant i \leqslant m$ we can find a path from $\langle x, v\rangle$ to $\langle u^{(i)}, v\rangle$; this path is of length at most $k+1$ and does not intersect any of the previously formed $d-1$ paths. Note that the node $\langle u^{(i)}, v\rangle$ and the node $\langle u, v\rangle$ are within the same Möbius cube and are adjacent each other for any $1 \leqslant i \leqslant m$. The above construction yields m node-disjoint paths from $\langle x = u, y\rangle$ to $\langle u, v\rangle$, which are of length at most $k+2$. The intermediate nodes in each of those m paths have different Möbius-part-labels, hence they do not intersect each other. Thus, there are totally $m+d-1$ node-disjoint paths from the source node $\langle x, y\rangle$ to the destination node $\langle u, v\rangle$, which are of length at most $k+2$.

Case 3: $x \neq u$ and $y \neq v$. Consider the deBruijn digraph $D(d,k)$; there are at least $d-1$ node-disjoint paths from node y to node v since one neighbor of y might be itself. These paths are of length at most k and are addressed as $dp_0, dp_1, \ldots, dp_{d-2}$.

The first m node-disjoint paths from node $\langle x, y\rangle$ to node $\langle u, v\rangle$ can be constructed as follows. Let Möbius-part-neighbors of the source node $\langle x, y\rangle$ are denoted as $\langle x^{(i)}, y\rangle$ for each $i \in \{1,\ldots,m\}$. Since nodes with Möbius-part-label $x^{(i)}$ for any $1 \leqslant i \leqslant m$ form a deBruijn digraph, proceeding from $\langle x^{(i)}, y\rangle$ we can reach $\langle x^{(i)}, v\rangle$ through any one, denoted as dp, of those $d-1$ paths from y to v within this deBruijn digraph, without loss of generality, let dp be dp_0. There exist m node-disjoint paths from node $\langle x, v\rangle$ to node $\langle u, v\rangle$ within the Möbius cube consisting of nodes with a deBruijn-part-label v. Since $\langle x^{(i)}, v\rangle$ for any $1 \leqslant i \leqslant m$ is the second node of each of such m pathes, we find m node-disjoint paths from the set of nodes $\langle x^{(i)}, v\rangle$ for all $1 \leqslant i \leqslant m$ to the node $\langle u, v\rangle$, which are of length at most d_1.

Consequently, the above construction yields m node-disjoint paths from the node $\langle x, y\rangle$ to the node $\langle u, v\rangle$, which are of length at most $d_1 + k + 1$. For example, node $\langle 00, 00\rangle$ can reach node $\langle 11, 11\rangle$ in Fig. 1(d) along the following $m = 2$ node-disjoint paths $\langle 00, 00\rangle \to \langle 10, 00\rangle \to \langle 10, 01\rangle \to \langle 10, 11\rangle \to \langle 11, 11\rangle$ and $\langle 00, 00\rangle \to \langle 01, 00\rangle \to \langle 01, 01\rangle \to \langle 01, 11\rangle \to \langle 11, 11\rangle$.

The other $d-1$ node-disjoint paths use a different construction approach. The d deBruijn-part-neighbors of the source node $\langle x, y\rangle$ is denoted as $\langle x, y^{(i)}\rangle$ for each $i \in \{0,\ldots,d-1\}$. Since one deBruijn-part-neighbor of the source node $\langle x, y\rangle$ might be itself, without loss of generality, let $\langle x, y^{(d-1)}\rangle$ be $\langle x, y\rangle$. Since nodes with a deBruijn-part-label $y^{(i)}$ for any $0 \leqslant i \leqslant d-2$ form a Möbius cube, proceeding from $\langle x, y^{(i)}\rangle$ we can reach $\langle u, y^{(i)}\rangle$ within this Möbius cube along a path. The deBruijn-part-label of each node in this path comes from dp_i from y to v in the deBruijn graph, while that of each node in the previously formed m paths from the node $\langle x, y\rangle$ to the node $\langle u, v\rangle$ comes from dp_0.

If $i \neq 0$, the resulting path from the node $\langle x, y\rangle$ to the node $\langle u, y^{(i)}\rangle$ does not intersect the previous m paths, due to the different deBruijn-part-labels. However, it is not true for $i = 0$ since the Möbius-part-neighbors of $\langle x, y^{(0)}\rangle$ have presented at the m paths. Inspired by Lemma 3, the node $\langle x, y^{(0)}\rangle$ can reach the node $\langle u, y^{(0)}\rangle$ along an alternative path, which traverses a cycle for node $y^{(0)}$ in a deBruijn digraph when only considering the deBruijn-part-labels of nodes in the alternative path. Let $y_1^{(0)}$ denote the second node in this cycle. Thus, node $\langle x, y^{(0)}\rangle$ reaches $\langle x, y_1^{(0)}\rangle$ in one hop, and $\langle u, y_1^{(0)}\rangle$ in at most d_1 additional hops within a Möbius cube, and finally $\langle u, y^{(0)}\rangle$ in at most k additional hops within a deBruijn digraph. For example, all Möbius-part-neighbors of $\langle x = 00, y^{(0)} = 01\rangle$ are $\langle 10, 01\rangle$ and $\langle 01, 01\rangle$, which have appeared at the aforementioned two paths from $\langle 00, 00\rangle$ to $\langle 11, 11\rangle$. Alternatively, we can reach $\langle u = 11, y^{(0)} = 01\rangle$ along a node-disjoint path $\langle 00, 01\rangle$, $\langle 00, 10\rangle$, $\langle 10, 10\rangle$, $\langle 11, 10\rangle$, and $\langle 11, 01\rangle$.

So far, we construct $d-1$ paths from $\langle x, y\rangle$ to $\langle u, y^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-2$, which do not intersect the existing m paths and are length of at most $d_1 + k + 1$. Since nodes $\langle u, y^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-2$ and node $\langle u, v\rangle$ are within the same deBruijn subgraph. Thus, we can prove that there are $d-1$ node-disjoint paths, which are of length at most k hops, from the set of nodes $\langle u, y^{(i)}\rangle$ for all $0 \leqslant i \leqslant d-2$ to the node $\langle u, v\rangle$ within that deBruijn subgraph, using the similar approach mentioned at the last paragraph in the proof of the case 1. In addition, such $d-1$ paths do not intersect the existing m paths from the node $\langle x, y\rangle$ to the node $\langle u, v\rangle$ since nodes in those pathes are different in the Möbius-part-labels. Thus, the above construction yields another $d-1$ node-disjoint paths from $\langle x, y\rangle$ to $\langle u, v\rangle$, which are of length $d_1 + k + 1 + k = d_1 + 2k + 1$. Recall that the previously formed m paths are of length at most $d_1 + k + 1$. Consequently, those $m + d - 1$ node-disjoint paths are of length at most $d_1 + 2k + 1$.

In summary, there are at least $m + d - 1$ node-disjoint paths from node $\langle x, y\rangle$ to node $\langle u, v\rangle$, which are of length at most $\max\{d_1 + 2k + 1, d_1 + k + 1, m\} = \max\{d_1 + 2k + 1, m\}$. Thus, Theorem 3 holds. □

Corollary 1. *The connectivity of the MD(m,d,k) is $m + d - 1$.*

3. Conclusion

This paper presents the Möbius–deBruijn, a new product graph of the Möbius cube and the deBruijn digraph, which employs the Möbius cube as a unit cluster and connects many such clusters by means of given number of parallel deBruijn digraphs. Möbius–deBruijn combines the advantages of both the two architectures, and also possesses the logarithmic diameter, the optimal connectivity, and the simple routing mechanism amenable to network faults. To the best of our knowledge, this is the first work dealing with the product of deBruijn digraph and Möbius cube. The basic ideas of the Möbius–deBruijn can apply to the product of deBruijn digraph and other hypercube-like networks, and also apply to the product of Kautz digraph and hypercube-like networks.

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